SOLUTIONS



MATHEMATICS SPECIALIST UNIT 3

Section Two:

Calculator-assumed

Semester 1, 2016

Student Number:	In Figures					
Student Number:	In Words					

Time allowed for this section

Reading time before commencing work: ten minutes

Working time for this section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler,

highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper,

and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	53	35
Section Two: Calculator-assumed	13	13	100	97	65
			Total	150	100

Instructions to candidates

- 1. The rules for the conduct of Western Australian external examinations are detailed in the Year 12 Information Handbook 2016. Sitting this examination implies that you agree to abide by these rules.
- 2. Write your answers in this Question/Answer Booklet.
- 3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.
- 4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
 - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
 - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.
 Fill in the number of the question that you are continuing to answer at the top of the page.
- 5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.
- 6. It is recommended that you **do not use pencil**, except in diagrams.
- 7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

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Section Two: Calculator-assumed

65% (97 marks)

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 (5 marks)

Consider the function $f(x) = x^2 - 4x$.

(a) Explain why it is necessary to restrict the natural domain of f in order that its inverse is also a function. (1 mark)

Solution					
f is not a one-to-one function for $x \in$					
Specific behaviours					
✓ explains function is not one-to-one over natural domain					

(b) State a minimal restriction to the domain of f that includes x = 3, and then use this restriction to determine $f^{-1}(x)$. (4 marks)

Solution f has a turning point at (2, -4) and so minimal restriction is $x \ge 2$ to include x = 3

$$y + 4 = x^2 - 4x + 4$$
$$= (x - 2)^2$$

$$x-2=\pm\sqrt{y+4}$$
 but choose $+\sqrt{}$ to ensure range of $f^{-1}=$ domain of f

$$x = 2 + \sqrt{y + 4}$$

$$f^{-1}(x) = 2 + \sqrt{x+4}$$

- \checkmark states minimal restriction to domain that includes x = 3
- √ completes square on RHS, adjusting LHS
- ✓ chooses, with reason, +ve root
- ✓ states inverse

Question 9 (5 marks)

(a) Let z be a non-zero complex number located in the complex plane. Describe the linear transformation(s) required to transform z to each of the following locations:

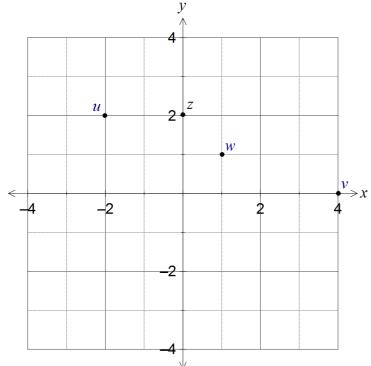
(i) 2z. (1 mark)

Solution				
Dilation of scale factor 2 about the origin.				
Specific behaviours				
✓ states dilation with scale factor and centre				

(ii) i^3z . (1 mark)

Solution				
Rotation of 270° anticlockwise about origin.				
Specific behaviours				
✓ states rotation with angle and centre				

(b) Consider the complex number z shown in the Argand diagram below. Add to the diagram the location of u, v and w where u=(1+i)z, $v=z\cdot \overline{z}$ and $w=\sqrt{z}$. (3 marks)



Solution				
See diagram				
Specific behaviours				
✓ places <i>u</i> correctly				
✓ places v correctly				
✓ places w correctly				

Question 10 (8 marks)

6

Two functions are given by $f(x) = 2\sqrt{x+1}$ and $g(x) = x^2 - 2x$.

(a) Determine g = f(x) and state the domain and range of this composite function. (3 marks)

Solution

$$g(f(x)) = (2\sqrt{x+1})^2 - 2(2\sqrt{x+1})$$
$$= 4(x+1) - 4\sqrt{x+1}$$
$$D_{gf} : x \in , x \ge -1$$
$$R_{gf} : y \in , y \ge -1$$

Specific behaviours

- ✓ writes simplified composite function
- √ states domain
- ✓ states range
- (b) Show that the composite function f(x) is defined for $x \in \mathbb{R}$. (3 marks)

Solution

$$f(g(x)) = 2\sqrt{x^2 - 2x + 1}$$

$$= 2\sqrt{(x - 1)^2}$$

$$= 2|x - 1|$$

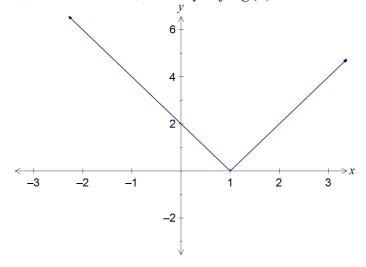
$$= \begin{cases} 2 - 2x, & x < 1 \\ 2x - 2, & x \ge 1 \end{cases}$$

Specific behaviours

- ✓ substitutes *g* into *f*
- √ simplifies root of square as absolute value
- ✓ shows piecewise definition for all real x

(c) Sketch the graph of y = f g(x) on the axes below.

(2 marks)



Solution

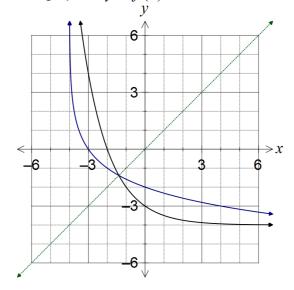
See diagram

- √ all axes intercepts
- √ shows two parts of composite function

Question 11 (12 marks)

7

(a) The graph of y = f(x) is shown below.



	Solution
(ii) See diagram	

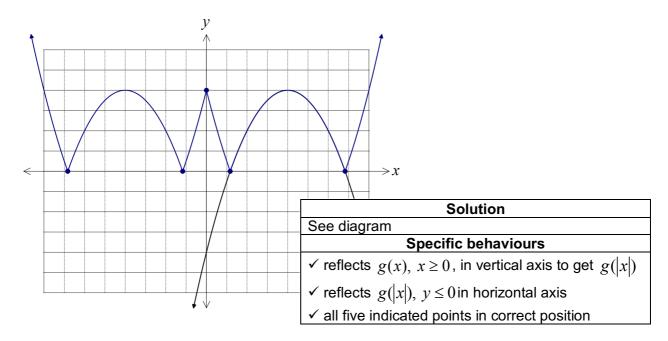
Specific behaviours

- ✓ both axes intercepts shown correctly
- ✓ curve approaches vertical asymptote x = -4
- ✓ smooth curve with y = x as mirror line

(i) What feature of the graph suggests that the inverse of f is a function? (1 mark)

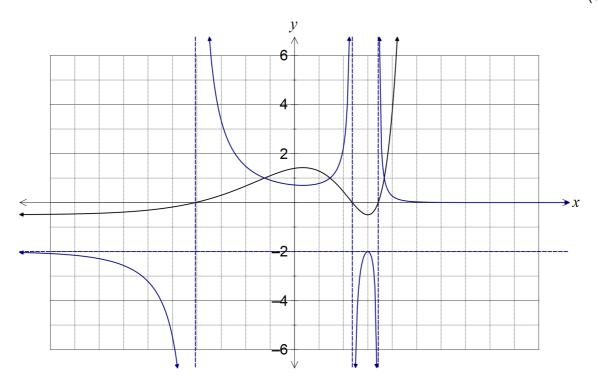
Solution				
The part of the graph shown is clearly one-to-one using the horizontal line test.				
Specific behaviours				
✓ describes function as one-to-one using graph feature				

- (ii) On the same axes, sketch the graph of the inverse of f, $y = f^{-1}(x)$. (3 marks)
- (b) The graph of y = g(x) is shown below. On the same axes, sketch the graph of y = |g(|x|)|. (3 marks)



(c) The graph of y=h(x) is shown below. As $x\to -\infty$, $h(x)\to -0.5$. On the same axes, sketch the graph of $y=\frac{1}{h(x)}$, clearly indicating all vertical and horizontal asymptotes.

(5 marks)



Solution				
See diagram				
Specific behaviours				
✓ correctly shows two parts of curve approaching horizontal asymptotes				
✓ correctly shows parts of curve approaching vertical asymptotes				
\checkmark correctly shows h and its reciprocal intersect three times when $y=1$				
✓ uses <i>y</i> -axis scale to locate min and max correctly				
✓ smooth curves used throughout				

Question 12 (8 marks)

(a) Determine all roots of the equation $z^6 + 8i = 0$, expressing them in exact polar form $rcis\theta$ where r > 0 and $-\pi < \theta \le \pi$. (5 marks)

Solution

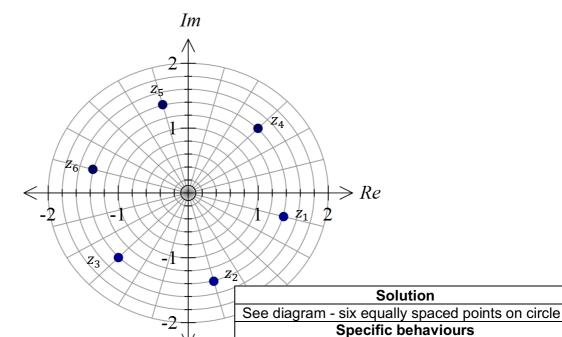
$$z^6 = 8cis\left(-\frac{\pi}{2}\right)$$

$$z = \sqrt[6]{8}cis\left(-\frac{\pi}{2} \times \frac{1}{6} + \frac{2\pi n}{6}\right), \ n = ..., -1, 0, 1, 2, ...$$
$$z = \sqrt{2}cis\left(-\frac{\pi}{12} + \frac{\pi n}{3}\right)$$

$$\begin{split} z_1 &= \sqrt{2}cis \left(-\frac{\pi}{12}\right), \ z_2 &= \sqrt{2}cis \left(-\frac{5\pi}{12}\right), \ z_3 &= \sqrt{2}cis \left(-\frac{3\pi}{4}\right) \\ z_4 &= \sqrt{2}cis \left(\frac{\pi}{4}\right), \ z_5 &= \sqrt{2}cis \left(\frac{7\pi}{12}\right), \ z_6 &= \sqrt{2}cis \left(\frac{11\pi}{12}\right) \end{split}$$

Specific behaviours

- √ expresses equation in polar form
- ✓ expresses first root with correct modulus
- ✓ expresses first root with correct argument
- √ determines argument between roots
- ✓ lists remaining five roots
- (b) Show all solutions of the equation on the Argand diagram below. (3 marks)



✓ locates roots with $r \approx 1.4$

✓ locates first root with correct argument

√ correctly spaces other five roots

See next page

Question 13 (7 marks)

A sphere is given by the equation $x^2 + y^2 + z^2 - 6x + 4y - 10z + 14 = 0$.

(a) Determine the vector equation of the sphere. (3 marks)

Completing the square:

$$x^{2} + y^{2} + z^{2} - 6x + 4y - 10z + 14 = 0$$

$$(x^{2} - 6x + 9) + (y^{2} + 4y + 4) + (z^{2} - 10z + 25) = -14 + 9 + 4 + 25$$

$$\therefore (x - 3)^{2} + (y + 2)^{2} + (z - 5)^{2} = 24$$

Hence, centre (3,-2,5) and radius $2\sqrt{6}$

$$\begin{vmatrix} \mathbf{r} - \begin{pmatrix} 3 \\ -2 \\ 5 \end{vmatrix} = 2\sqrt{6}$$

(b) Show that the line $\mathbf{r}(t) = (2t - 3)\mathbf{i} + (t - 8)\mathbf{j} + (11 - 4t)\mathbf{k}$ is tangent to the sphere; hence determine coordinates of the point of intersection.

(4 marks)

Intersection given by
$$\begin{vmatrix} 2t-3 \\ t-8 \\ 11-4t \end{vmatrix} - \begin{pmatrix} 3 \\ -2 \\ 5 \end{vmatrix} = 2\sqrt{6}$$

$$\begin{vmatrix} 2t - 6 \\ t - 6 \\ 6 - 4t \end{vmatrix} = 2\sqrt{6}$$

$$\sqrt{(2t-6)^2 + (t-6)^2 + (6-4t)^2} = 2\sqrt{6}$$
CAS:
$$\therefore 21t^2 - 84t + 84 = 0$$

$$21(t-2)^2 = 0$$

∴ t = 2 is a unique solution, which implies that the line intersects the circle only once, hence it is a tangent

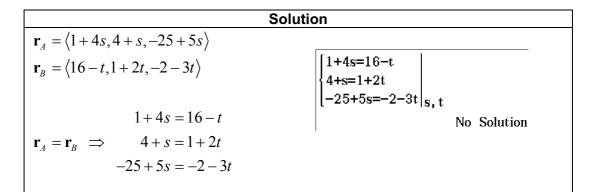
Point of intersection is i - 6j + 3k

Question 14 (7 marks)

11

Two small bodies, A and B, simultaneously leave their initial positions of $\mathbf{i} + 4\mathbf{j} - 25\mathbf{k}$ and $16\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, and move with constant velocities of $4\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ respectively.

(a) Determine whether the paths of the bodies cross or if the bodies meet. (4 marks)



System has no solution

Paths of bodies do not cross, so bodies do not meet.

- √ describe paths as vector equations
- ✓ equate coefficients
- ✓ states equations inconsistent/have no solutions
- ✓ interprets that paths do not cross
- (b) At the same time, a third small body, C, leaves its initial position. In its subsequent motion body C passes through the origin before crossing the path of body A. If C moves with a steady velocity of $5a\mathbf{i} + 5\mathbf{j} + a\mathbf{k}$, determine the value of the constant a. (3 marks)

Solution
$$\mathbf{r}_C = t \langle 5a, 5, a \rangle \text{ at time } t \text{ after pass thru O}$$

$$1 + 4s = 5at$$

$$\mathbf{r}_A = \mathbf{r}_C \implies 4 + s = 5t$$

$$-25 + 5s = at$$

$$t = 2, \ s = 6, \ a = 2.5$$
Specific behaviours

- √ describe path of C as vector equation
- √ equate coefficients
- ✓ states value of a is 2.5

Question 15 (8 marks)

12

The function f is defined by $f(x) = \frac{x^2 - 1}{x^2 - 3x + 2}$.

(a) Determine the natural domain and range of f(x).

(3 marks)

Solution $f(x) = \frac{(x+1)(x-1)}{(x-2)(x-1)}$ $= \frac{(x+1)}{(x-2)}, x \neq 1 \Rightarrow \text{Range}: y \in , x \neq 1, x \neq 2$ $= 1 + \frac{3}{x-2}, x \neq 1$ Specific behaviours

- \checkmark factorises and simplifies f
- √ states domain
- ✓ states range using asymptote

(b) Show that the function has no stationary points.

shows cannot be zero

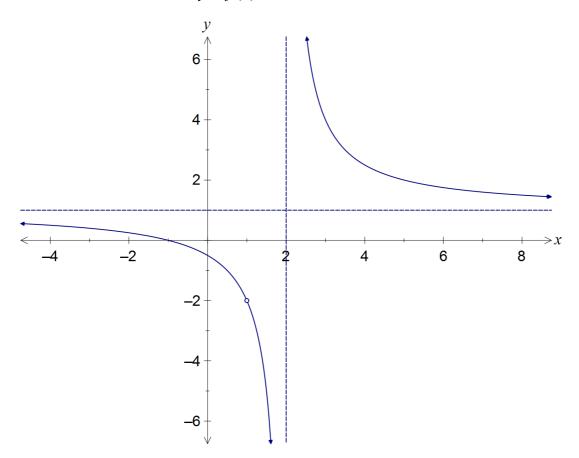
(2 marks)

Solution
$$f'(x) = \frac{-3}{(x-2)^2}$$

$$-3 \neq 0 \implies f'(x) \neq 0$$
Specific behaviours
$$\checkmark \text{ states } f'(x)$$

(c) Sketch the graph of y = f(x) on the axes below.

(3 marks)



Solution				
See diagram				
Specific behaviours				
√ axes intercepts				
✓ smooth curves approach asymptotes correctly				
√ indicates undefined point				

Question 16 (8 marks)

Given the two complex numbers $w = r(\cos\theta + i\sin\theta)$ and $z = s(\cos\phi + i\sin\phi)$, determine the following in terms of the non-zero constants r, s, θ and ϕ :

(a)
$$arg(\bar{z})$$
. (1 mark)

(b)
$$\left| \frac{i}{w^2} \right|$$
. (2 marks)

$$\frac{i}{w^2} = \frac{cis\left(\frac{\pi}{2}\right)}{r^2 \cdot cis2\theta} = \frac{1}{r^2}cis\left(\frac{\pi}{2} - 2\theta\right) \implies \left|\frac{i}{w^2}\right| = \frac{1}{r^2}$$
Specific behaviours

- ✓ simplifies into acisb form
- √ states modulus

(c)
$$|(1-i)wz|$$
. (2 marks)

Solution
$$(1-i)wz = \sqrt{2}cis\left(-\frac{\pi}{4}\right) \cdot r \cdot cis\theta \cdot s \cdot cis\phi = \sqrt{2}rs \cdot cis\left(\theta + \phi - \frac{\pi}{4}\right)$$

$$|(1-i)wz| = \sqrt{2}rs$$
 Specific behaviours

- √ simplifies into acisb form
- ✓ determines modulus

(d)
$$\arg\left(-\frac{z}{iw}\right)$$
. (3 marks)

$$\frac{\text{Solution}}{-\frac{z}{iw}} = \frac{cis(\pi) \cdot s \cdot cis\phi}{cis(\frac{\pi}{2}) \cdot r \cdot cis\theta} = \frac{s}{r} \cdot cis(\frac{\pi}{2} + \phi - \theta)$$

$$\arg(-\frac{z}{iw}) = \frac{\pi}{2} + \phi - \theta$$
Specific behaviours

- ✓ writes -1 and i in cis form
- ✓ simplifies into acisb form
- ✓ determines argument

Question 17 (7 marks)

Consider the three vectors $\mathbf{a} = \langle 2, 1, -3 \rangle$, $\mathbf{b} = \langle -3, 5, -2 \rangle$ and $\mathbf{c} = \langle 2, -4, 1 \rangle$.

(a) Prove that the three vectors do not lie in the same plane.

(4 marks)

Solution

If vectors lie in the same plane, then a vector perpendicular to ${\bf a}$ and ${\bf b}$ will also be perpendicular to ${\bf c}$.

Vector perpendicular to **a** and **b** is **d**:

$$\mathbf{d} = \langle 2, 1, -3 \rangle \times \langle -3, 5, -2 \rangle$$
$$= \langle 13, 13, 13 \rangle$$

Consider scalar product of **c** and **d**:

$$\langle 2, -4, 1 \rangle \langle 13, 13, 13 \rangle = -13$$

Since this is not zero, then **c** and **d** are not perpendicular, and so we conclude that the three vectors cannot lie in the same plane.

Specific behaviours

- ✓ chooses cross product to find a perpendicular
- √ calculates perpendicular correctly
- ✓ chooses scalar product to show perpendicular not perpendicular to other vector
- ✓ shows scalar product is not 0
- (b) Determine the value(s) of the constant a if the vector $\langle a^2, a, a-3 \rangle$ lies in the same plane as vectors \mathbf{a} and \mathbf{b} . (3 marks)

Solution

$$\langle 1,1,1\rangle \langle a^2,a,a-3\rangle = a^2 + a + a - 3$$

$$a^{2} + 2a - 3 = 0$$

 $(a+3)(a-1) = 0 \implies a = -3, 1$

- √ calculates scalar product
- √ solves scalar product equal to zero
- √ determines all values of a

Question 18 (6 marks)

16

The curve shown has the vector equation

$$\mathbf{r}(t) = \left(\frac{2\cos(t)}{\sin^2(t) + 1}\right) \mathbf{i} + \left(\frac{4\sin(t)\cos(t)}{\sin^2(t) + 1}\right) \mathbf{j}$$

Show algebraically that the Cartesian equation of this curve can be written in the format

$$(4x^2 + y^2)^a = b(4x^2 - y^2)$$

and hence state the value of a and b.

(6 marks)

Solution

$$y = 2 \sin(t) \left(\frac{2 \cos(t)}{\sin^2(t) + 1} \right) = 2x \sin(t)$$

$$\therefore \sin(t) = \frac{y}{2x} \text{ and hence } \sin^2(t) = \frac{y^2}{4x^2} \text{ and } \cos(t) = \frac{\sqrt{4x^2 - y^2}}{2x}$$

$$\therefore x = \frac{2 \cos(t)}{\sin^2(t) + 1} = \frac{2\left(\frac{\sqrt{4x^2 - y^2}}{2x}\right)}{\frac{y^2}{4x^2} + 1}$$

$$x\left(\frac{y^2 + 4x^2}{4x^2}\right) = \frac{1}{x}\sqrt{4x^2 - y^2}$$

$$4x^2 + y^2 = 4\sqrt{4x^2 - y^2}$$

$$\therefore (4x^2 + y^2)^2 = 16(4x^2 - y^2)$$
Hence, $a = 2$ and $b = 16$

Question 19 (9 marks)

17

Let the complex number $z = \cos \theta + i \sin \theta$.

(a) Show that
$$\frac{1}{z} = \cos \theta - i \sin \theta$$
. (2 marks)

Solution
$$\frac{1}{z} = z^{-1}$$

$$= cis(-\theta)$$

$$= cos(-\theta) + i sin(-\theta)$$

$$= cos \theta - i sin \theta$$
Specific behaviours

✓ uses De Moivre's theorem to obtain $cis(-\theta)$

✓ uses trig identity to obtain result

(b) Show that
$$z^3 - \frac{1}{z^3} = 2i\sin 3\theta$$
. (2 marks)

Solution
$$z^{3} - \frac{1}{z^{3}} = cis(3\theta) - cis(-3\theta)$$

$$= \cos 3\theta + i \sin 3\theta - \cos 3\theta + i \sin 3\theta$$

$$= 2i \sin 3\theta$$
Specific behaviours

✓ uses De Moivre's theorem to obtain triple angles

√ simplifies result

(c) Determine
$$\operatorname{Im}\left(z^3 - \frac{1}{z^3}\right)$$
 in terms of $\sin\theta$ and $\cos\theta$.

(3 marks)

Solution
$$z^{3} = \cos^{3}\theta + 3i\cos^{2}\theta\sin\theta + 3i^{2}\cos\theta\sin^{2}\theta + i^{3}\sin^{3}\theta$$

$$= \cos^{3}\theta + 3i\cos^{2}\theta\sin\theta - 3\cos\theta\sin^{2}\theta - i\sin^{3}\theta$$

$$\frac{1}{z^{3}} = \cos^{3}\theta - 3i\cos^{2}\theta\sin\theta + 3i^{2}\cos\theta\sin^{2}\theta - i^{3}\sin^{3}\theta$$

$$= \cos^{3}\theta - 3i\cos^{2}\theta\sin\theta - 3\cos\theta\sin^{2}\theta + i\sin^{3}\theta$$

$$Im\left(z^{3} - \frac{1}{z^{3}}\right) = 3\cos^{2}\theta\sin\theta - \sin^{3}\theta - \left(-3\cos^{2}\theta\sin\theta + \sin^{3}\theta\right)$$

$$= 6\cos^{2}\theta\sin\theta - 2\sin^{3}\theta$$
Specific behaviours

✓ expands z^3

✓ expands z^{-3}

√ simplifies imaginary part

(d) Express $\sin^3 \theta$ in terms of $\sin \theta$ and $\sin 3\theta$.

(2 marks)

Solution

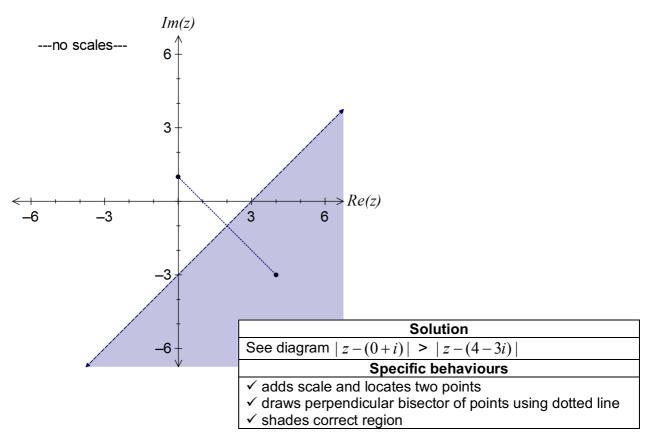
Using results from (b) and (c):

$$\operatorname{Im}\left(z^{3} - \frac{1}{z^{3}}\right) = 2\sin 3\theta = 6\cos^{2}\theta\sin\theta - 2\sin^{3}\theta$$
$$2\sin 3\theta = 6(1 - \sin^{2}\theta)\sin\theta - 2\sin^{3}\theta$$
$$\sin 3\theta = 3\sin\theta - 4\sin^{3}\theta$$
$$\sin^{3}\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

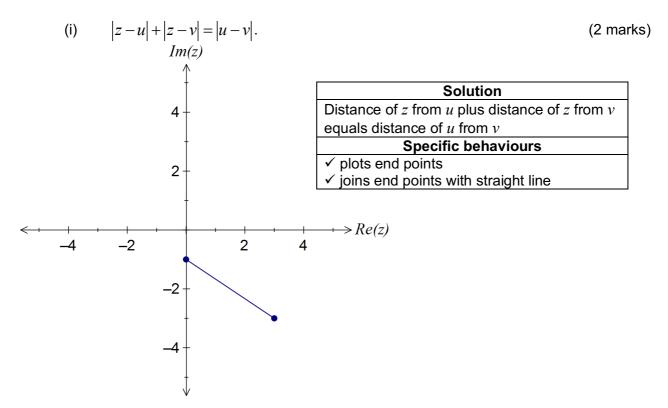
- \checkmark equates and eliminates $\cos\theta$ from results in (b) and (c)
- √ simplifies and rearranges for required result

Question 20 (7 marks)

(a) Shade the region satisfying the complex inequality |z-i| > |z-4+3i| on the Argand diagram below. (3 marks

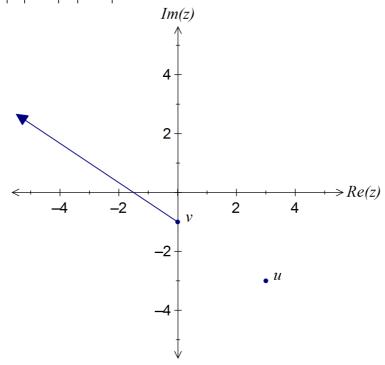


(b) Consider the two complex numbers given by u = 3 - 3i and v = -i. Sketch each of the following sets of points in the complex plane.



(ii)
$$|z-v|+|u-v|=|z-u|$$
.

(2 marks)



Solution

Distance of z from v plus distance of u from v equals distance of z from u

- ✓ constructs straight line using points
- \checkmark indicates solution extends from v

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MATHEMATICS SPECIALIST UNIT 3

Additional working space

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Question	number:	
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